

A DIVERGENCE THEOREM FOR NON-COMPACT RIEMANNIAN MANIFOLDS: A DYNAMICAL APPROACH

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ABSTRACT. In this paper we use a dynamical approach to prove some new divergence theorems on complete non-compact Riemannian manifolds.

1. Introduction and statement of results

A divergence theorem states that $\int_M (\operatorname{div} X) d\nu_g = 0$, under certain assumptions on X and M , where M is a Riemannian manifold, X is a vector field on M and $\operatorname{div} X$ denotes the divergence of X . The starting point is the usual divergence theorem for the case where X is smooth and has compact support.

On a closed Riemannian manifold the classical divergence theorem is a very useful tool in the study of PDEs. In particular, in the study of differential operators in divergence form, such as the Laplace-Beltrami operator, the p -Laplace operator and the mean curvature operator.

In this context, the divergence theorem can be used to obtain an integration by parts that facilitates the obtaining of integral estimates of solutions of PDEs. Furthermore, it is possible to use the divergence theorem to obtain comparison results and deduce information of the geometry and the topology of this Riemannian manifold.

If a divergence theorem is available for a complete non-compact Riemannian manifold then it would be interesting to generalize some of the existing results for compact Riemannian manifolds.

In that direction, we present a new divergence theorem for the non-compact case, with or without finite volume, based on the dynamical behavior of the geodesic flow and the integrability of a functional associated to the vector field. To precise, let us introduce some definitions.

Let (M, g) be a complete Riemannian manifold without boundary, we denote by ∇ the Levi-Civita connection. Given a C^1 vector field on M we can consider the continuous function f_X on the unit tangent bundle SM defined by $f_X(p, v) = g(\nabla_v X, v)$.

In the theorem below we present a new divergence theorem on complete Riemannian manifolds without boundary whose geodesic flow is recurrent.

THEOREM 1.1. *Let M be a non-compact complete Riemannian manifold without boundary and X a C^1 vector field. If the geodesic flow ϕ_t is recurrent with respect to the Liouville measure and f_X is integrable on SM then $\operatorname{div} X$ is integrable on M and*

$$\int_M \operatorname{div} X d\nu_g = 0.$$

In particular, if M has finite volume we get the following corollary.

COROLLARY 1.2. *Let M be a non-compact complete Riemannian manifold without boundary with finite volume and X a C^1 vector field on M . If f_X is integrable on SM then $\operatorname{div} X$ is integrable on M and*

$$\int_M \operatorname{div} X \, d\nu_g = 0.$$

If (M, g) is a non-compact Riemannian manifold whose geodesic flow is not necessarily recurrent then using the E. Hopf's decomposition and an additional hypothesis we get the following result.

THEOREM 1.3. *Let M be a non-compact complete Riemannian manifold without boundary and X a C^1 vector field. If f_X is integrable on SM and $|X| \rightarrow 0$ uniformly at infinity in M then $\operatorname{div} X$ is integrable on M and*

$$\int_M \operatorname{div} X \, d\nu_g = 0.$$

The essential tool in the proofs is the Maximal Ergodic Theorem (see section 2.5); similar approach was already used in [5] and [19].

It is worth mentioning that there exist Riemannian manifolds with infinite volume whose geodesic flow is recurrent with respect to Liouville measure; an example of this type of manifold is the hyperbolic surface of divergence type.

A hyperbolic surface is a complete, two dimensional Riemannian manifold with constant curvature -1 . This surface has the unit disc as universal cover and can be viewed as H/Γ where H is the unit disc equipped with the hyperbolic metric and Γ is the covering group of isometries of H .

In 1939, Hopf, proved that geodesic flows on hyperbolic surfaces of infinite area are either totally dissipative or recurrent and ergodic. Nicholls (see [10]) proved that the geodesic flow on the hyperbolic surface H/Γ is recurrent and ergodic if and only if the Poincaré series of Γ diverges at $s = 1$, this result was also proved by Aaronson and Sullivan in [2]. The hyperbolic surfaces whose Poincaré series of Γ diverges at $s = 1$ are called of divergence type.

Other type of examples go as follows: given a closed subset Λ of the Riemann sphere $\hat{\mathbb{C}}$ containing at least 3 points; it is known that the Riemann surface $M = \hat{\mathbb{C}} \setminus \Lambda$ has universal cover \mathbb{H}^2 , where \mathbb{H}^2 is the hyperbolic space and therefore admits a hyperbolic metric. One knows that the hyperbolic area of M is infinite once Λ is infinite. Furthermore, if Λ has logarithmic capacity zero for every positive measure μ on Λ then the geodesic flow on M is ergodic and hence M is a hyperbolic surface of divergence type. For more details see [2].

So we have many examples of hyperbolic surfaces with infinite volume whose geodesic flow is ergodic and recurrent.

In [1], Denker and Aaronson also construct other examples of Riemannian manifolds with infinite volume whose geodesic flow is recurrent.

For the case of zero curvature, the geodesic flow associated to “infinite staircase square tiled surface (which provide a non-compact surface with infinite volume and planar curvature) it is also recurrent (see for instance [7]).

Similar type of divergences results were already obtained. In [3], Gaffney proved that if X is a C^1 vector field such that $|X| \in L^1(M)$ and $\operatorname{div} X$ has an integral (i.e either $(\operatorname{div} X)^+$ or $(\operatorname{div} X)^-$ is integrable) then $\int_M \operatorname{div} X \, d\nu_g = 0$. This result was generalized by Karp in [8] with a weaker hypothesis about the norm of the vector field X . Similar results for the non-complete case were also obtained for p -parabolic Riemannian manifolds (see details in section 2.4).

The main difference of the above mentioned results with our new divergence theorems is that we assume an integrability condition on the function f_X but not necessarily in the

norm of the vector field, which is the common hypothesis in the results proved somewhere else.

In fact, using either surfaces of revolution or the classical construction of warped product, we prove in section 6 that there are examples (with finite and infinite volume) that satisfy that f_X is integrable but $|X|$ is not integrable. For that it is used proposition 5.1 (see section 5) that provides a sufficient condition under which the function f_X is integrable (it is useful to compare the statement of above mentioned proposition with the hypothesis of the theorem due by Karp (see theorem 2.7).

At the end of the paper, we provide some applications to potential theory.

2. Notation and Preliminaries

In this section we present the notations and some preliminaries results that will be used in the sequel. Throughout the rest of the paper, (M, g) will denote a Riemannian manifold without boundary with dimension $n \geq 2$.

2.1. Fubini's Theorem for the case of Riemannian submersions. Let ν_g be the Riemannian measure or volume measure associated to Riemannian manifold (M, g) . Now consider a Riemannian submersion $\pi : (M, g) \rightarrow (N, h)$. Recall that for each $q \in \pi(M)$, $\pi^{-1}(q)$ is an $(n - m)$ -dimensional submanifold, which carries the Riemannian measure ν_{g_q} with respect to the metric g_q induced on $\pi^{-1}(q)$ from g . For a function f defined over M , we set f_q the restriction of f to $\pi^{-1}(q)$. Now if f_q is an integrable function on $\pi^{-1}(q)$ with respect to ν_{g_q} , define

$$\bar{f}(q) = \int_{\pi^{-1}(q)} f_q d\nu_{g_q}.$$

We conclude the subsection stating a result that will be used in the next section.

THEOREM 2.1. (*Theorem 5.6, p. 66, [16]*) *Let $\pi : (M, g) \rightarrow (N, h)$ be a surjective Riemannian submersion. If f is a real-valued continuous function with compact support (resp., an integrable function) on M , then \bar{f} is a continuous function with compact support on N (resp., f_q is integrable for almost all $q \in N$ and \bar{f} is a integrable function on N), and*

$$\int_M f d\nu_g = \int_N \bar{f} d\nu_h.$$

For more details see [16].

2.2. Geodesic flow. Let (M, g) be a complete Riemannian manifold. We denote by $\gamma_\theta(t)$ the unique geodesic with initial conditions $\gamma_\theta(0) = p$ and $\gamma'_\theta(0) = v$, where $\theta = (p, v)$ is a point in the tangent bundle TM . For a given $t \in \mathbb{R}$, we define the following diffeomorphism, $\phi_t : TM \rightarrow TM$ with $\phi_t(\theta) = (\gamma_\theta(t), \gamma'_\theta(t))$.

The family of diffeomorphism ϕ_t is in fact a flow (called *geodesic flow*) once it satisfies $\phi_{t+s} = \phi_t \circ \phi_s$.

Let SM be the unit tangent bundle of M , i.e, the subset of TM given by those pairs $\theta = (p, v)$ such that v has norm one. Since geodesics travel with constant speed, we see that ϕ_t leaves SM invariant, that is, given $\theta \in SM$ then for every $t \in \mathbb{R}$ we have that $\phi_t(\theta) \in SM$.

Let $\pi : TM \rightarrow M$ be the canonical projection, where $\pi(\theta) = p$.

DEFINITION 2.2. There exists a canonical subbundle of TTM called the vertical subbundle whose fiber at θ is given by $V(\theta) = \ker(d_\theta \pi)$.

We shall define the connection map $K : TTM \rightarrow TM$ as follows. Let $\xi \in T_\theta TM$ and $z : (-\epsilon, \epsilon) \rightarrow TM$ be a curve adapted to ξ , i.e, $z(0) = \theta$ and $z'(0) = \xi$, where $z(t) = (\alpha(t), Z(t))$,

$$K_\theta(\xi) = \frac{DZ}{dt}(0).$$

The map K_θ is well defined and is linear for each θ .

DEFINITION 2.3. The horizontal subbundle is the subbundle of TTM whose fiber at θ is given by $H(\theta) = \ker(K_\theta)$.

For each θ , the maps $d_\theta\pi|_{H(\theta)} : H(\theta) \rightarrow T_pM$ and $K_\theta|_{V(\theta)} : V(\theta) \rightarrow T_pM$ are linear isomorphisms. Furthermore, $T_\theta TM = H(\theta) \oplus V(\theta)$ and the map $j_\theta : T_\theta TM \rightarrow T_pM \times T_pM$ given by

$$j_\theta(\xi) = (d_\theta\pi(\xi), K_\theta(\xi)),$$

is a linear isomorphism.

DEFINITION 2.4. Using the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$, we can define in a natural way a Riemannian metric on TM that makes $H(\theta)$ and $V(\theta)$ orthogonal. This metric is called the Sasaki metric and is given by

$$g_\theta^S(\xi, \eta) = g_{\pi(\theta)}(d_\theta\pi(\xi), d_\theta\pi(\eta)) + g_{\pi(\theta)}(K_\theta(\xi), K_\theta(\eta))$$

Now consider the Sasaki metric restricted to the unit tangent bundle SM . The projection $\pi : SM \rightarrow M$ is a surjective Riemannian submersion. Furthermore, the geodesic flow in SM preserves the Riemannian measure ν_{g^S} induced by the Sasaki metric. This measure coincides with the Liouville measure up to a constant. When M has finite volume the Liouville measure is finite. For more details see [12].

2.3. Warped products. Let B, F be Riemannian manifolds, with metrics g_B and g_F , respectively and $f > 0$ be a smooth function on B . The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the Riemannian metric

$$g = \pi_B^*(g_B) + (f \circ \pi_B)^2 \pi_F^*(g_F),$$

where π_B and π_F are the projections of $B \times F$ onto B and F , respectively.

Let X be a vector field on B . The horizontal lift of X to $B \times_f F$ is the vector field \overline{X} such that $d\pi_{B(p,q)}(\overline{X}(p,q)) = X(p)$ and $d\pi_{F(p,q)}(\overline{X}(p,q)) = 0$. If Y is a vector field on F , the vertical lift of Y to $B \times F$ is the vector field \overline{Y} such that $d\pi_{B(p,q)}(\overline{Y}(p,q)) = 0$ and $d\pi_{F(p,q)}(\overline{Y}(p,q)) = Y(q)$. The set of all such lifts are denoted as usual by $\mathcal{L}(B)$ and $\mathcal{L}(F)$, respectively.

We denote by ∇, ∇^B and ∇^F the Levi-Civita connections on $B \times_f F$, B and F , respectively.

PROPOSITION 2.5. (*Proposition 35, p. 206, [11]*) On $M = B \times_f F$, if $\overline{X}, \overline{Y} \in \mathcal{L}(B)$ and $\overline{U}, \overline{V} \in \mathcal{L}(F)$ then

- (1) $\nabla_{\overline{X}} \overline{Y} = \overline{\nabla_X^B Y}$,
- (2) $\nabla_{\overline{U}} \overline{X} = (Xf/f) \overline{U}$,
- (3) $d\pi_B(\nabla_{\overline{U}} \overline{V}) = -(g(U, V)/f) \cdot \text{grad } f$,
- (4) $d\pi_F(\nabla_{\overline{U}} \overline{V}) = \nabla_U^F V$.

2.4. The Divergence theorem. In this subsection we present some results that can be found in the literature on divergence theorems.

Let X be a C^1 vector field on a Riemannian manifold (M, g) .

DEFINITION 2.6. The differential of X is the linear operator $A_X : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, given by $A_X(Y) := \nabla_Y X$. Then, to each point $p \in M$, we assign the linear map $A_X(p) : T_p M \rightarrow T_p M$ defined by $A_X(p)v = \nabla_v X$. The divergence of X , denoted by $\operatorname{div} X$, is the trace of this differential.

If X has compact support and M is a Riemannian manifold then from classic divergence theorem it follows that

$$\int_M \operatorname{div} X \, d\nu_g = 0.$$

Gaffney in [3], extended this result to complete Riemannian manifolds M by proving that, given a C^1 vector field X on M , we have

$$\int_M \operatorname{div} X \, d\nu_g = 0,$$

provided $|X|$ and $\operatorname{div} X$ are integrable (but in fact, if $(\operatorname{div} X)^- = \max\{\operatorname{div} X, 0\}$ is integrable then the result remains true). This result was later extended by Karp. He proved the following result,

THEOREM 2.7. ([8]) *Let M^n be a complete non-compact Riemannian manifold without boundary and X a vector field such that*

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \int_{B(2r)/B(r)} |X| \, d\nu_g = 0.$$

If $\operatorname{div} X$ has an integral then

$$\int_M \operatorname{div} X \, d\nu_g = 0.$$

In particular, if outside of some compact set $\operatorname{div} X$ is everywhere ≥ 0 (or ≤ 0) then

$$\int_M \operatorname{div} X \, d\nu_g = 0.$$

REMARK 2.8. In the previous theorem, $B(r)$ denotes the geodesic ball of radius r at some point $p \in M$.

To consider the non-necessary complete manifolds let us introduce the notion of p -parabolicity.

DEFINITION 2.9. A Riemannian manifold M is said to be p -parabolic, $1 < p < \infty$, if every solution $u \in W_{loc}^{1,p}(M) \cap C^0(M)$ of the problem

$$\begin{cases} \Delta_p u \geq 0 & \text{on } M, \\ \sup_M u < +\infty \end{cases}$$

must be constant. A manifold that is not p -parabolic will be called p -hyperbolic.

There are several equivalent definitions of this notion, for more details see [14], [17]. In the definition above $W_{loc}^{1,p}(M)$ denotes the space of the local L^p -functions such that for every compact K the distributional gradient are in $L^p(K)$ and $\Delta_p u$ denotes the non-linear operator defined by,

$$\Delta_p u = \operatorname{div} (|\operatorname{grad} u|^{p-2} \cdot \operatorname{grad} u).$$

This operator is usually called p -Laplace operator. When $p = 2$ this operator coincides with the Laplace-Beltrami operator.

A complete Riemannian manifold with finite volume is p -parabolic for all $1 < p < \infty$. A complete Riemannian manifold with polynomial growth of degree d is p -parabolic for all $p \geq d$. For example a complete n -dimensional Riemannian manifold with non negative Ricci curvature is p -parabolic for all $p \geq n$, in particular the Euclidean space \mathbb{R}^n is p -parabolic for all $p \geq n$.

A complete simply connected Riemannian manifold with sectional curvature $K \leq -1$ is p -hyperbolic for all $1 < p < \infty$, in particular the hyperbolic space \mathbb{H}^n is p -hyperbolic for all $1 < p < \infty$.

Let M be a Riemannian manifold 2-parabolic, not necessarily complete. From the results of Lyons and Sullivan in [9], it follows that if a vector field X satisfies $|X| \in L^1(M)$, $\operatorname{div} X \in L^1_{\operatorname{loc}}(M)$ and $(\operatorname{div} X)^-$ is integrable then $\int_M \operatorname{div} X d\nu_g = 0$. In the general case, from the results of Gol'dshtein and Troyanov in [4] with the assumption that $|X| \in L^{\frac{p}{p-1}}$, it follows that this result remains true for p -parabolic Riemannian manifolds with $p > 1$ thus gives us a p -parabolic analogue of the Gaffney result. Moreover, also in [4] it is proved that a Riemannian manifold M is p -parabolic if, and only if, the following result holds. Let X be a vector field satisfying $|X| \in L^{\frac{p}{p-1}}$, $\operatorname{div} X \in L^1_{\operatorname{loc}}(M)$ and $\operatorname{div} X$ has an integral. Then

$$\int_M \operatorname{div} X d\nu_g = 0.$$

In [18] Valtorta and Veronelli present some new Stokes's theorem on complete manifolds that extends, in different directions, previous works of Gaffney and Karp and also the so called Kelvin-Nevanlinna-Royden criterion for p -parabolicity.

2.5. Ergodic Theory. We say that (X, \mathcal{B}) is a standard measurable space if X is a Polish space and \mathcal{B} is the Borel σ -algebra. Let (X, \mathcal{B}, m) be standard measure space, that is, (X, \mathcal{B}) is a standard measurable space. We say that $A = B \pmod{m}$ if $m((A \setminus B) \cup (B \setminus A)) = 0$.

A measurable map $T : X \rightarrow X$ is called measure preserving if $m(C) = m(T^{-1}(C))$ for all $C \in \mathcal{B}$.

DEFINITION 2.10. A continuous flow $\{\phi_t\}_{t \in \mathbb{R}}$ on X preserves a measure m if each ϕ_t is a measure preserving transformation with respect to m .

We say that a set $Z \subset X$ is ϕ_t -invariant if for all $t \in \mathbb{R}$, $\phi_t(Z) = Z \pmod{m}$.

Now we state the Maximal ergodic theorem that will be used in the proof of the main result of this paper. This result can be found in [13].

THEOREM 2.11. (*Maximal ergodic theorem*) Let M be a complete Riemannian manifold equipped with the σ -algebra \mathcal{B} of Borel sets, a flow $\{\phi_t\}_{t \in \mathbb{R}}$ and a invariant measure for the flow μ . If f is a measurable function such that f^+ or f^- is integrable and $Z \subset M$ is a ϕ_t -invariant Borel set then

$$\int_{Z_f} f d\mu \geq 0,$$

where $Z_f = \{x \in Z \mid \sup_{s > 0} \int_0^s f(\phi_t(x)) dt > 0\}$.

Observe that in the hypothesis of the Maximal Ergodic Theorem the Riemannian manifold M does not necessarily has finite volume.

We say that $p \in X$ is a recurrent point for a continuous flow $\{\phi_t\}_{t \in \mathbb{R}}$ on X if there exists a sequence $t_j \rightarrow +\infty$ in \mathbb{N} such that $\phi_{t_j}(p) \rightarrow p$.

DEFINITION 2.12. We say that a flow $\{\phi_t\}_{t \in \mathbb{R}}$ that preserves the measure m is recurrent if, given any measurable set A , for m -a.e $x \in A$ there exists a sequence $t_n \rightarrow \infty$ such that $\phi_{t_n}(x) \in A$.

2.6. The decomposition of E.Hopf and a measure on the spaces of orbits.

Following [5], let $f_0 > 0$ be an integrable function on SM and $\{\phi_t\}_{t \in \mathbb{R}}$ the geodesic flow. Then the Borel sets

$$D^+ = \left\{ \theta \in SM : \int_0^\infty f_0(\phi_t(\theta)) dt < \infty \right\}, \quad C^+ = SM \setminus D^+$$

are ϕ_t -invariant. Furthermore, they are independent of f_0 in the following sense: if $f_1 \geq 0$ is another integrable function, then $\nu_g^S(E) = 0$, where

$$E = \left\{ \theta \in SM : \int_0^\infty f_1(\phi_t(\theta)) dt = \infty \right\} \cap D^+.$$

The decomposition $SM = D^+ \cup C^+$ is called E. Hopf's decomposition of SM associated to the geodesic flow $\{\phi_t\}_{t \in \mathbb{R}}$. The components D^+ and C^+ are called, respectively, the dissipative and the conservative parts of the decomposition. Denote by $SM = D^- \cup C^-$ the decomposition of E. Hopf of SM associated to the inverse flow $\{\phi_{-t}\}_{t \in \mathbb{R}}$. Then

$$D^- = \left\{ \theta \in SM : \int_{-\infty}^0 f_0(\phi_t(\theta)) dt < \infty \right\}.$$

From the above discussions, if f is an integrable function on SM then for almost all $\theta \in D = D^+ \cap D^-$, the Lebesgue integral

$$(2.1) \quad \int_{-\infty}^\infty f(\phi_t(\theta)) dt < \infty,$$

exists. Let \sim be the equivalence relation in SM defined by: $\theta \sim \eta$ if and only if there exists $j \in \mathbb{Z}$ such that $\phi_j(\theta) = \eta$. Let SM/\sim be the space of orbits, and denote by $\pi : SM \rightarrow SM/\sim$ the natural projection $\pi(\theta) = [\theta]$. Consider in SM/\sim the σ -algebra $\tilde{\beta}$ induced by π . Then $\pi(E)$ is measurable for every Borel set E . A Borel set E is called a wandering set if $\phi_j(E) \cap E = \emptyset$ for every $j \geq 1$. For each $\tilde{E} \in \tilde{\beta}$, let

$$\tilde{\mu}(\tilde{E}) = \sup\{\nu_g^S(E) : E \subset \pi^{-1}(\tilde{E}), \text{ } E \text{ is a wandering set}\}.$$

Guimarães proved in [5] that $\tilde{\mu}$ is a measure on SM/\sim with the property that $\tilde{\mu}(\pi(E)) = \mu(E)$ for every wandering set. Given a function f integrable on SM , then by (2.1), $[\theta] \mapsto \int_{-\infty}^\infty f(\phi_t(\theta)) dt$, $[\theta] \in \pi(D)$, defines a $\tilde{\mu}$ -measurable function on $\pi(D)$.

PROPOSITION 2.13. (*Proposition 2.3, [5]*) *If f is an integrable function on SM , then*

$$\int_D f d\nu_g^S = \int_{\pi(D)} \int_{-\infty}^\infty f(\phi_t(\theta)) dt d\tilde{\mu}.$$

3. Proof of Theorem 1.1

We denote by ν_g^S the Riemannian measure on SM induced by Sasaki metric. In the next lemma we relate the integral of the function f_X on SM with the integral of the divergence of X on M .

LEMMA 3.1. *Let $(f_X)_p$ be the restriction of f_X to $\pi^{-1}(p)$. Then*

$$\overline{f_X}(p) = \int_{\pi^{-1}(p)} (f_X)_p d\nu_{g_p}^S = \frac{\omega_{n-1}}{n} \cdot \operatorname{div} X(p),$$

where ω_{n-1} is the volume of the $n-1$ dimensional sphere in \mathbb{R}^n with the canonical metric.

PROOF. Fix an orthonormal basis $\{e_1, \dots, e_n\}$ in $T_p M$ and consider the charts $\varphi_+ : U_+ \rightarrow B$ and $\varphi_- : U_- \rightarrow B$, where

$$B = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : x_1^2 + \dots + x_{n-1}^2 < 1\},$$

$U_+ = \{w \in \pi^{-1}(p) : g(w, e_n) > 0\}$ and $U_- = \{w \in \pi^{-1}(p) : g(w, e_n) < 0\}$. The maps φ_+ and φ_- are defined by

$$\varphi_+^{-1}(x_1, \dots, x_{n-1}) = \sum_{i=1}^n x_i e_i \quad \text{and} \quad \varphi_-^{-1}(y_1, \dots, y_{n-1}) = \sum_{i=1}^n y_i e_i,$$

with $x_n = \sqrt{1 - (x_1^2 + \dots + x_{n-1}^2)}$ and $y_n = -\sqrt{1 - (y_1^2 + \dots + y_{n-1}^2)}$. Therefore,

$$\begin{aligned} \overline{f_X}(p) &= \int_B (\sqrt{G^+} \circ \varphi_+^{-1}(x)) \left(\sum_{i,j=1}^n x_i x_j g(\nabla_{e_i} X, e_j) \right) dx_1 \cdots dx_{n-1} \\ &\quad + \int_B (\sqrt{G^-} \circ \varphi_-^{-1}(y)) \left(\sum_{i,j=1}^n y_i y_j g(\nabla_{e_i} X, e_j) \right) dy_1 \cdots dy_{n-1}, \end{aligned}$$

where G^+ denotes the determinant of a matrix (g_{ij}^+) consisting of the components of g with respect to local coordinates (x_i) and G^- denotes the determinant of a matrix (g_{ij}^-) consisting of the components of g with respect to the local coordinates (y_i) . On the other hand, if $i \neq j$ and $i, j \leq n-1$ we have,

$$\int_B x_i x_j (\sqrt{G^+} \circ \varphi_+^{-1}(x)) dx_1 \cdots dx_{n-1} = \int_{\mathbb{S}_+^{n-1}} z_i z_j d\mathbb{S}^{n-1} = 0,$$

where $\mathbb{S}_+^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{S}^{n-1} : x_n > 0\}$. For $i \leq n-1$ we have,

$$\int_B x_i x_n (\sqrt{G^+} \circ \varphi_+^{-1}(x)) dx_1 \cdots dx_{n-1} = \int_{\mathbb{S}_+^{n-1}} z_i \sqrt{1 - (z_1^2 + \dots + z_{n-1}^2)} d\mathbb{S}^{n-1} = 0,$$

and

$$\int_B x_i^2 (\sqrt{G^+} \circ \varphi_+^{-1}(x)) dx_1 \cdots dx_{n-1} = \int_{\mathbb{S}_+^{n-1}} z_i^2 d\mathbb{S}^{n-1} = \frac{\omega_{n-1}}{2n}.$$

If $i = n$ we have,

$$\begin{aligned} \int_B x_n^2 (\sqrt{G^+} \circ \varphi_+^{-1}(x)) dx_1 \cdots dx_{n-1} &= \int_{\mathbb{S}_+^{n-1}} 1 - (z_1^2 + \dots + z_{n-1}^2) d\mathbb{S}^{n-1} \\ &= \frac{\omega_{n-1}}{2} - \frac{n-1}{2n} \omega_{n-1} \\ &= \frac{\omega_{n-1}}{2n}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_B (\sqrt{G^+} \circ \varphi_+^{-1}(x)) \left(\sum_{i,j=1}^n x_i x_j g(\nabla_{e_i} X, e_j) \right) dx_1 \cdots dx_{n-1} &= \frac{\omega_{n-1}}{2n} \sum_{i=1}^n g(\nabla_{e_i} X, e_i) \\ &= \frac{\omega_{n-1}}{2n} \operatorname{div} X(p). \end{aligned}$$

Proceeding in the same way, we get

$$\int_B (\sqrt{G^-} \circ \varphi_-^{-1}(y)) \left(\sum_{i,j=1}^n y_i y_j g(\nabla_{e_i} X, e_j) \right) dy_1 \cdots dy_{n-1} = \frac{\omega_{n-1}}{2n} \operatorname{div} X(p).$$

Hence,

$$\overline{f_X}(p) = \int_{\pi^{-1}(p)} (f_X)_p d\nu_{g_p^S} = \frac{\omega_{n-1}}{n} \cdot \operatorname{div} X(p).$$

□

Fix an integrable function h on M strictly positive such that $\int_0^1 h(\gamma_\eta(t)) dt > 0$ for every $\eta \in SM$. Now consider a recurrent point $\theta = (p, v)$ in SM and the function \tilde{h} on SM where $\tilde{h}(p, v) = h(p)$. We have

$$\begin{aligned} \int_0^t (f_X + \tilde{h})(\phi_t(\theta)) ds &= \int_0^t \frac{d}{ds} g(X(\gamma_\theta(s)), \gamma'_\theta(s)) ds + \int_0^t h(\gamma_\theta(s)) dt \\ &= g(X(\gamma_\theta(t)), \gamma'_\theta(t)) - g(X(p), v) + \int_0^t h(\gamma_\theta(s)) dt. \end{aligned}$$

Since θ is recurrent there exists a sequence $t_n \rightarrow +\infty$ such that $\phi_{t_n}(\theta) \rightarrow \theta$ thus $g(X(\gamma'_\theta(t_n)), \gamma'_\theta(t_n)) \rightarrow g(X(p), v)$. Hence, $\sup_{t>0} \int_0^t (f_X + \tilde{h})(\phi_s(x)) ds > 0$. Since the geodesic flow is recurrent, for ν_g^S -a.e $\theta \in SM$ the point θ is recurrent. Applying the Maximal ergodic theorem, we conclude that

$$\int_{SM} (f_X + \tilde{h}) d\nu_{g^S} \geq 0.$$

Repeating this argument with the function $1/k \cdot h$, it follows that for every $n \in \mathbb{N}$,

$$\int_{SM} f_X d\nu_{g^S} \geq -\frac{1}{k} \int_{SM} \tilde{h} d\nu_{g^S}$$

Hence,

$$\int_{SM} f_X d\nu_{g^S} \geq 0.$$

On the other hand, the function f_{-X} is also integrable on SM . Applying the argument above, it follows that

$$\int_{SM} f_X d\nu_{g^S} \leq 0.$$

Therefore, $\int_{SM} f_X d\nu_{g^S} = 0$. From Theorem 2.1 and Lemma 3.1, we have that $\operatorname{div} X$ is integrable and

$$\begin{aligned} \int_M \operatorname{div} X d\nu_g &= \frac{n}{\omega_{n-1}} \int_{SM} f_X d\nu_{g^S}. \\ &= 0. \end{aligned}$$

This concludes the proof of Theorem 1.1. If (M, g) has finite volume then from Poincaré recurrence theorem it follows that the geodesic flow ϕ_t is recurrent. As an immediate consequence of Theorem 1.1, we get the Corollary 1.2. The Maximal ergodic theorem was also used by Guimarães in [5] to get a rigidity result on Riemannian manifolds without conjugate points.

4. Proof of Theorem 1.3

Observe that SM is the disjoint union $SM = D \cup C^+ \cup (C^- \setminus C^+)$. Since C^+ is ϕ_t -invariant and for almost all $\theta \in C^+$, θ is recurrent, by the proof of Theorem 1.1 we have

$$\int_{C^+} f_X d\nu_{g^S} = 0.$$

Note that $C^- \setminus C^+$ is also ϕ_t -invariant and for almost all $\theta \in C^-$, θ is recurrent with respect to the inverse geodesic flow. By the proof of Theorem 1.1 we have

$$\int_{(C^- \setminus C^+)} f_X d\nu_{g^S} = 0.$$

Let d, \tilde{d} be the distances on M and SM respectively, for almost all $\theta \in D$ we have $\tilde{d}(\theta, \phi_t(\theta)) \rightarrow \infty$ as $|t| \rightarrow \infty$. On the other hand, we have $\tilde{d}(\theta, \phi_t(\theta)) \leq d(p, \gamma_\theta(t)) + 2\pi$, where $\theta = (p, v)$. Thus, for almost all $\theta \in SM$ we get $d(p, \gamma_\theta(t)) \rightarrow \infty$ as $|t| \rightarrow \infty$.

On the other hand,

$$\left| \int_{-s}^s f_X(\phi_t(\theta)) dt \right| \leq |X(\gamma_\theta(s))| + |X(\gamma_\theta(-s))|.$$

Since f_X is integrable and $|X| \rightarrow 0$ at infinity in M , for almost all $\theta \in D$ it follows that

$$\int_{-\infty}^{\infty} f_X(\phi_t(\theta)) dt = 0.$$

By Proposition 2.13 follows that

$$\int_D f_X d\nu_g^S = 0.$$

Therefore,

$$\begin{aligned} \int_{SM} f_X d\nu_g^S &= \int_D f_X d\nu_g^S + \int_{C^+} f_X d\nu_g^S + \int_{(C^- \setminus C^+)} f_X d\nu_g^S \\ &= 0. \end{aligned}$$

From Theorem 2.1 and Lemma 3.1, we have that $\operatorname{div} X$ is integrable and

$$\begin{aligned} \int_M \operatorname{div} X d\nu_g &= \frac{n}{\omega_{n-1}} \int_{SM} f_X d\nu_g^S \\ &= 0. \end{aligned}$$

5. Sufficient conditions for the integrability of f_X

Here we present a proposition that guarantee the integrability of f_X which is used in the examples in the next section.

PROPOSITION 5.1. *Let M be a complete non-compact Riemannian manifold without boundary and X a C^1 vector field such that*

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \int_{B(2r) \setminus B(r)} |X| d\nu_g < \infty.$$

If f_X has an integral (i.e either f_X^+ or f_X^- is integrable) then f_X is integrable.

Without loss of generality, we may assume that f_X^- is integrable. It is known (see [19]) that there is a constant $C > 0$ such that for each $r > 0$ exists a continuous function φ_r satisfying: $0 \leq \varphi_r \leq 1$, $\varphi_r \equiv 1$ on $B(r)$, $\varphi_r \equiv 0$ on the complement of $B(2r)$ and

$\|\text{grad } \varphi_r(x)\| \leq C/r$. Consider the function defined by $f_r(p, v) = \varphi_r(p)f_X(p, v)$. Since f_r has a compact support it follows from Proposition 3.1 and Theorem 2.1 that

$$\int_{SM} f_r d\nu_{g^S} = \frac{\omega_{n-1}}{n} \int_M \varphi_r \cdot \text{div } X d\nu_g.$$

On the other hand, we have that $\text{div}(\varphi_r X) = \varphi_r \cdot \text{div } X + g(\text{grad } \varphi_r, X)$. From the classic divergence theorem it follows that

$$\int_M \varphi_r \cdot \text{div } X d\nu_g = - \int_M g(\text{grad } \varphi_r, X) d\nu_g.$$

Hence,

$$\left| \int_M (\varphi_r \cdot \text{div } X) d\nu_g \right| \leq \frac{C}{r} \int_{B(2r)/B(r)} |X| d\nu_g.$$

Consider the set defined by $A_r = \{(p, v) : p \in B(2r)\}$. We have

$$\begin{aligned} \int_{A_r} f_r^+ d\nu_{g^S} - \int_{SM} f_X^- d\nu_{g^S} &\leq \left| \int_{SM} f_r d\nu_{g^S} \right| \\ &\leq \frac{C \cdot \omega_{n-1}}{nr} \int_{B(2r)/B(r)} |X| d\nu_g \end{aligned}$$

We may choose $r_i \rightarrow +\infty$ such that the right-hand side of this inequality is bounded. Thus f_X^+ is also integrable and therefore f_X is integrable.

6. Examples

In this section we present four examples related to the hypothesis we assume and we discuss their relation with the existing results; more precisely, we provide different type of examples satisfying our hypothesis which are not covered by previous results.

In the first example we construct a smooth vector field X that satisfies the hypothesis of Theorem 2.7, f_X is integrable on SM but $|X|$ is non-integrable on M .

In the second example we construct a complete non-compact Riemannian manifold with finite volume M_α^3 and a smooth vector field \bar{Z} that does not satisfy the hypothesis of Theorem 2.7, but f_X is integrable on SM . Furthermore, $|\bar{Z}|^p$ is not integrable for every $p \geq 1$. This example shows that the condition of integrability of the function f_X is really different of the sufficient conditions used in [3], [8], [4] and [18] to ensure that the integral of the divergence is null.

Furthermore, in the third example we construct a complete Riemannian manifold with infinite volume M and a smooth vector field \bar{U} that satisfies the hypothesis of Theorem 1.3 but does not satisfy the sufficient conditions used in [3], [8], [4] and [18] to ensure that the integral of the divergence is null.

When the Riemannian manifold is compact the integral of the divergence of a C^1 vector field X is zero. An interesting question is to know whether this phenomenon also occurs in non-compact Riemannian manifold with finite volume for the vector fields whose divergence is integrable. In the last example we will answer this question by constructing an example of a complete Riemannian manifold non-compact with finite volume N and a smooth vector field Z on M such that $\text{div } Z$ is integrable on M , but

$$\int_M \text{div } X d\nu_g > 0.$$

EXAMPLE 6.1. Let $N \subset \mathbb{R}^3$ be the surface obtained by rotating the graph of the function $f(x) = \frac{1}{1+x^2}$, where $x \in \mathbb{R}$, around the axis x with the usual metric. Then, S is a complete non-compact surface.

We will show below that S has finite area. We have

$$\nu_g(N) = 2\pi \int_{-\infty}^{+\infty} \frac{1}{1+x^2} \sqrt{1 + \frac{4x^2}{(1+x^2)^4}} dx = 4\pi \int_0^{+\infty} \frac{1}{1+x^2} \sqrt{1 + \frac{4x^2}{(1+x^2)^4}} dx.$$

Since $\lim_{x \rightarrow +\infty} \frac{4x^2}{(1+x^2)^4} = 0$ there exists $x_0 \in \mathbb{R}$ such that $\frac{4x^2}{(1+x^2)^4} \leq 3$, for all $x \geq x_0$. Therefore,

$$\nu_g(N) \leq 4\pi \int_0^{x_0} \frac{1}{1+x^2} \sqrt{1 + \frac{4x^2}{(1+x^2)^4}} dx + 8\pi \int_{x_0}^{+\infty} \frac{1}{1+x^2} dx < +\infty.$$

We can also write $N = G^{-1}(\{0\})$ where $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ is given by

$$G(x, y, z) = y^2 + z^2 - \frac{1}{(1+x^2)^2}.$$

Hence, if $p = (x, y, z) \in M$ then

$$T_p N = \left\{ (a, b, c) : g\left((a, b, c), \left(\frac{4x}{(1+x^2)^3}, 2y, 2z\right)\right) = 0 \right\}.$$

Consider the smooth vector field $W(x, y, z) = x(1+x^2) \cdot (0, -z, y)$. The function $|W|$ is non-integrable because

$$\int_N |W| d\nu_g = \lim_{r \rightarrow +\infty} \int_0^{2\pi} \int_{-r}^r \frac{|s|}{1+s^2} \sqrt{1 + \frac{4s^2}{(1+s^2)^4}} ds dt = +\infty.$$

On the other hand, given $v = (a, b, c) \in T_p N$ with $a^2 + b^2 + c^2 = 1$ follows that

$$\begin{aligned} f_W(p, v) &= g(\nabla_v W, v) \\ &= a(1+3x^2)(-zb + cy). \end{aligned}$$

Hence, $\operatorname{div} W = 0$ and

$$\begin{aligned} |f_W(p, v)| &\leq (1+3x^2) \sqrt{b^2 + c^2} \sqrt{z^2 + y^2} \\ &\leq \frac{1+3x^2}{1+x^2} \\ &\leq 3. \end{aligned}$$

Therefore f_W is integrable on SN .

Now consider the points $p = (0, 1, 0)$, $q = (x, y, z)$ and $q_1 = \left(x, \frac{1}{1+x^2}, 0\right)$ in M . Let d be the distance on M . We have

$$\sqrt{x^2 + (y-1)^2 + z^2} \leq d(p, q) \leq d(p, q_1) + d(q_1, q).$$

Observe that there exists $C > 1$ such that

$$|x| \leq d(p, q) \leq C|x| + \frac{\pi}{1+x^2} \leq C|x| + \pi.$$

For each r consider the set F_r defined by,

$$F_r = \{(x, y, z) \in M : r < |x| \leq 2Cr + 2\pi\}.$$

From the above inequalities it follows that $B(2Cr + 2\pi)/B(Cr + \pi) \subset F_r$. Hence, for r large we get

$$\begin{aligned} \int_{B(2Cr+2\pi)/B(Cr+\pi)} |W| d\nu_g &\leq \int_{F_r} |W| d\nu_g \\ &= 4\pi \int_r^{2Cr+2\pi} \frac{|s|}{1+s^2} \sqrt{1 + \frac{4s^2}{(1+s^2)^4}} ds \\ &\leq 8\pi \int_r^{2Cr+2\pi} \frac{|s|}{1+s^2} ds \\ &= 4\pi(\log(2Cr + 2\pi) - \log r). \end{aligned}$$

Therefore,

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \int_{B(2r)/B(r)} |W| d\nu_g \leq \lim_{r \rightarrow +\infty} \frac{4\pi(\log(2Cr + 2\pi) - \log r)}{r} = 0.$$

EXAMPLE 6.2. Let (M, g) be the warped product $\mathbb{H}^2 \times_h \mathbb{S}^1$ where \mathbb{H}^2 is the hyperbolic plane and \mathbb{S}^1 is the unit circle. Fix a point $(y, a) \in M$ and consider the real function $h : \mathbb{H}^2 \rightarrow (0, \infty)$ defined by $h(p) = b(d(p, y))$, where d denotes the distance in \mathbb{H}^2 and $b : \mathbb{R} \rightarrow (0, \infty)$ is a C^∞ function such that $b(r) = a > 0$ if $|r| < 1$,

$$\lim_{r \rightarrow +\infty} b(r) = 0, \int_0^\infty b(r) \cdot \sinh r \, dr < \infty \text{ and } \lim_{r \rightarrow +\infty} (\sinh r)^2 \cdot b(r) > 0.$$

Now consider the parametrization $\Psi : (0, \infty) \times (0, 2\pi) \times (0, 2\pi) \rightarrow M$, given by $\Psi(r, \theta, t) = (\varphi(r, \theta), \eta(t))$, where $\varphi(r, \theta) = \exp_y(r\gamma(\theta))$, $\gamma(\theta)$ is a circle of radius 1 in $T_y\mathbb{H}^2$ with center 0 parametrized by the central angle θ and $\eta(t)$ is the parametrization of \mathbb{S}^1 by the central angle. We have,

$$\nu_g(M) = 4\pi^2 \int_0^\infty \sinh r \cdot b(r) \, dr.$$

Therefore (M, g) is a complete non-compact Riemannian manifold with finite volume. Now consider the unique smooth vector field Z on \mathbb{H}^2 such that $Z(\exp_y(r\gamma(\theta))) = \frac{\partial \varphi}{\partial \theta}(r, \theta)$ for every $(r, \theta) \in (0, \infty) \times (0, 2\pi)$. From Koszul formula it follows that Z is a Killing vector field.

On the other hand, from Proposition 2.5 it follows that the horizontal lift of Z to M is also a Killing vector field. Observe that,

$$||\bar{Z}(\Psi(r, \theta, t))|| = \left| \frac{\partial \varphi}{\partial \theta}(r, \theta) \right| = \sinh r,$$

and

$$\int_M |\bar{Z}|^p d\nu_g = 4\pi^2 \int_0^\infty (\sinh r)^{p+1} \cdot (b(r)) \, dr.$$

Thus $|\bar{Z}|^p$ is not integrable for every $p \geq 1$. Let \bar{d} be the distance on M . Given $(p, b) \in M$ notice that

$$d(y, p) \leq \bar{d}((y, a), (p, b)) \leq \bar{d}((y, a), (p, a)) + \bar{d}((p, a), (p, b)) \leq d(y, p) + 2\pi b(d(p, y)).$$

Let $C = \sup_{r \geq 0} b(r)$ and for each $R > \max\{4\pi C, 3\}$, let A_R be the set defined by

$$A_R = \{(p, b) \in M_\alpha^3 : R < d(y, p) < \frac{3}{2}R\}.$$

It follows from the above inequalities that $A_R \subset B(3/2R)/B(R)$, where $B(R)$ denotes the geodesic ball of radius R and center (y, a) . Hence,

$$\begin{aligned} \int_{B(2R)/B(R)} |\bar{Z}| d\nu_g &\geq \int_{A_R} |\bar{Z}| d\nu_g \\ &= 4\pi^2 \int_R^{3R/2} (\sinh r)^2 \cdot b(r) dr. \end{aligned}$$

Which implies that

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \int_{B(2R)/B(R)} |\bar{Z}| d\nu_g > 0.$$

EXAMPLE 6.3. Let (M, g) be the warped product $\mathbb{H}^2 \times_h \mathbb{S}^1$ where \mathbb{H}^2 is the hyperbolic plane and \mathbb{S}^1 is the unit circle. Fix a point $(y, a) \in M$ and consider the real function $b : \mathbb{H}^2 \rightarrow (0, \infty)$ defined by $b = b(d(p, y))$, where d denotes the distance in \mathbb{H}^2 and $b : \mathbb{R} \rightarrow (0, \infty)$ is a C^∞ function such that $b(r) = a > 0$ if $|r| < 1$, $\lim_{r \rightarrow +\infty} b(r) = 0$ and $\lim_{r \rightarrow +\infty} \sinh r \cdot (b(r))^p > 0$,

for every $p \geq 1$. Now consider a smooth vector field U on \mathbb{S}^1 such that $\langle U, U \rangle = 1$ with respect to usual metric of \mathbb{S}^1 . Following the same notation of the previous example, by definition we have that $|\bar{U}(\Psi(r, \theta, t))| = b(r)$, where \bar{U} denotes the vertical lift of U to M . From Proposition 2.5 it follows that \bar{U} is a Killing vector field on M thus $f_{\bar{U}}$ is integrable on SM . This shows that the vector field \bar{U} satisfies the hypothesis of Theorem 1.3.

On the other hand, proceeding in the same way of the previous example we have

$$\nu_g(M) = 4\pi^2 \int_0^\infty \sinh r \cdot b(r) dr,$$

and

$$\int_M |\bar{U}|^p d\nu_g = 4\pi^2 \int_0^\infty \sinh r \cdot (b(r))^{p+1} dr.$$

Since $\lim_{r \rightarrow +\infty} \sinh r \cdot (b(r))^q > 0$ for every $q \geq 1$ it follows that the Riemannian manifold M has infinite volume and that $|\bar{U}|^q$ is not integrable for every $q \geq 1$. By the discussion of the previous example we have

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \int_{B(2R)/B(R)} |\bar{U}| d\nu_g \geq \lim_{R \rightarrow +\infty} \frac{4\pi^2}{R} \int_R^{3R/2} \sinh r \cdot (b(r))^2 dr.$$

Therefore,

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \int_{B(2R)/B(R)} |\bar{U}| d\nu_g > 0.$$

EXAMPLE 6.4. Let \mathbb{L}^3 be the 3-dimensional Lorentz space, that is, the real vector space \mathbb{R}^3 endowed with the Lorentzian metric

$$\langle p, q \rangle = p_1 q_1 + p_2 q_2 - p_3 q_3.$$

The 2-dimensional hyperbolic space

$$\mathbb{H}^2 = \{p \in \mathbb{L}^3; \langle p, p \rangle = -1, p_3 \geq 1\},$$

as it is well known, is a space like hypersurface in \mathbb{L}^3 , that is, the induced metric via the inclusion $\iota : \mathbb{H}^2 \hookrightarrow \mathbb{L}^3$ is a Riemannian metric on \mathbb{H}^2 . Consider the hyperbolic space \mathbb{H}^2 with the orientation given by the normal vector field $N(x, y, z) = (-x, -y, -z)$. Let $X \in \mathfrak{X}(\mathbb{H}^2)$

be the vector field defined as the projection of the vector $-e_3$ on each tangent plane of \mathbb{H}^2 , i.e.,

$$X(x, y, z) = -e_3 + \langle -e_3, N(x, y, z) \rangle N(x, y, z) = (xz, yz, z^2 - 1).$$

It is not difficult to see that X is a conformal vector field with conformal factor $\psi(x, y, z) = z > 0$, that is, $\langle \nabla_V X, U \rangle + \langle \nabla_U X, V \rangle = 2\psi \langle V, U \rangle$, where U and V are vector fields on M and ∇ is the Levi-Civita connection on \mathbb{H}^2 .

Let (M, g) be the warped product $\mathbb{H}^2 \times_f \mathbb{S}^1$, where the function $f : \mathbb{H}^2 \rightarrow \mathbb{R}$ is given by $f(x, y, z) = 1/z^2$. Now consider the parametrization $\Psi : \mathbb{R}^2 \times (0, 2\pi) \rightarrow M$, given by $\Psi(x, y, t) = (\varphi(x, y), \eta(t))$, where $\varphi(x, y) = (x, y, \sqrt{1 + x^2 + y^2})$ and $\eta(t)$ is the parametrization of \mathbb{S}^1 by the central angle. Observe that,

$$\begin{aligned} \nu_g(M) &= 2\pi \int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + y^2)^{3/2}} dx dy \\ &= 4\pi^2 \int_0^{+\infty} \frac{r}{(1 + r^2)^{3/2}} dr \\ &= 4\pi^2. \end{aligned}$$

Therefore M is a complete Riemannian manifold, non-compact with finite volume. Let Z be the horizontal lift of X to M . From Proposition 2.5, it follows that

$$\operatorname{div} Z = \operatorname{div} X + Xf/f.$$

On the other hand, we can write $X = xz \frac{\partial \varphi}{\partial x} + yz \frac{\partial \varphi}{\partial y}$. Hence

$$Xf = -\frac{2z(x^2 + y^2)}{(1 + x^2 + y^2)^2}.$$

Therefore,

$$\operatorname{div} Z = \frac{2}{\sqrt{1 + x^2 + y^2}}.$$

The $\operatorname{div} Z$ is integrable on M . In fact, we have

$$\begin{aligned} \int_M \operatorname{div} Z d\nu_g &= 2\pi \int_{\mathbb{R}^2} \frac{2}{(1 + x^2 + y^2)^2} dx dy \\ &= 4\pi^2 \int_0^{+\infty} \frac{2r}{(1 + r^2)^2} dr \\ &= 4\pi^2. \end{aligned}$$

7. Applications

In this subsection we present some applications related to potential theory. According to the terminology introduced by Rigoli-Setti in [15]. The φ -Laplacian of a function u is the nonlinear, divergence form operator defined by

$$L_\varphi(u) = \operatorname{div} (|\operatorname{grad} u|^{-1} \varphi(|\operatorname{grad} u|) \cdot \operatorname{grad} u)$$

where $\varphi \in C^0([0, +\infty)) \cap C^1((0, +\infty))$ satisfies the following structural conditions:

- (1) $\varphi(0) = 0$,
- (2) $\varphi(t) > 0 \ \forall t > 0$,
- (3) $\varphi(t) \leq At^{r-1}$, for some constants $A, r > 1$.

If $\varphi(t) = t^{p-1}$, $1 < p < +\infty$, the φ -Laplacian corresponds to the usual p -Laplace operator

$$\Delta_p u = \operatorname{div}(|\operatorname{grad} u|^{p-2} \cdot \operatorname{grad} u).$$

In particular, if $p = 2$, the p -Laplace operator is nothing but the Laplace-Beltrami operator.

When $\varphi(t) = \frac{t}{\sqrt{1+t^2}}$, the φ -Laplacian corresponds to the mean curvature operator

$$H_u = \operatorname{div}\left(\frac{\operatorname{grad} u}{\sqrt{1 + |\operatorname{grad} u|^2}}\right).$$

For more details see [14]. In the theorem below we demonstrate a global comparison principle for complete Riemannian manifolds with infinite volume whose geodesic flow is recurrent. Similar results can be found in [18], [14] and [6].

In [6] the authors proved a global comparison result for the p -Laplacian on a p -parabolic manifold and in [14] we also can find results on a p -parabolic manifold. In particular, for closed manifolds and complete Riemannian manifolds with finite volume. It is worth mentioning that the next theorem can be applied to some hyperbolic surfaces with infinite volume and these manifolds are not p -parabolic.

LEMMA 7.1. (*Lemma 1.14, [14]*) *If φ is strictly increasing and satisfies the above assumptions then the following holds. Let $(V, \langle \cdot, \cdot \rangle)$ be an n -dimensional, real vector space endowed with the scalar product $\langle \cdot, \cdot \rangle$. Then, for every $\xi, \eta \in V$,*

$$h(\xi, \eta) = \langle |\xi|^{-1} \varphi(|\xi|) \xi - |\eta|^{-1} \varphi(|\eta|) \eta, \xi - \eta \rangle \geq 0,$$

with equality holding if and only if $\xi = \eta$.

THEOREM 7.2. *Assume that (M, g) is a complete Riemannian manifold with infinite volume whose geodesic flow is recurrent with respect to Liouville measure. Consider the φ -Laplace operator L_φ , where φ is strictly increasing and satisfies the above assumptions. If $u, v \in C^2(M)$ are solutions of $L_\varphi(u) \geq L_\varphi(v)$, f_X, f_Y are integrable on SM where $X = |\operatorname{grad} u|^{-1} \varphi(|\operatorname{grad} u|) \cdot \operatorname{grad} u$, $Y = |\operatorname{grad} v|^{-1} \varphi(|\operatorname{grad} v|) \cdot \operatorname{grad} v$ and $|\operatorname{grad} u|, |\operatorname{grad} v| \in L^r(M)$ then $u - v \equiv \text{const.}$*

PROOF. Fix any $x_0 \in M$. Let $A = u(x_0) - v(x_0)$ and define Ω_A to be the connected component of the open set $\{x \in M : A - 1 < u(x) - v(x) < A + 1\}$ which contains x_0 . Let $\alpha : \mathbb{R} \rightarrow [0, +\infty)$ be a smooth function that satisfies the following conditions:

- (1) $\alpha \equiv 0$ if $t \leq A - 1$,
- (2) $\alpha \equiv 1$ if $t \geq A + 1$,
- (3) $\alpha' > 0$ if $A - 1 < t < A + 1$,

Now consider, the vector field $Z = \alpha \circ (u - v) \cdot (X - Y)$. We have

$$f_Z(p, w) = \alpha' \circ (u - v) g(\operatorname{grad} u - \operatorname{grad} v, w) g(X - Y, w) + \alpha \circ (u - v) (f_X(p, w) - f_Y(p, w)).$$

Observe that

$$\begin{aligned} |f_Z| &\leq C \cdot |\operatorname{grad} u - \operatorname{grad} v| |X - Y| + |f_X| + |f_Y| \\ &\leq C \cdot (|\operatorname{grad} u| + |\operatorname{grad} v|) (|X| + |Y|) + |f_X| + |f_Y| \\ &\leq C \cdot A (|\operatorname{grad} u|^r + |\operatorname{grad} v|^r + |\operatorname{grad} v| |\operatorname{grad} u|^{r-1} + |\operatorname{grad} u| |\operatorname{grad} v|^{r-1}) \\ &\quad + |f_X| + |f_Y|. \end{aligned}$$

By Young's inequality we get

$$|\operatorname{grad} u| |\operatorname{grad} v|^{r-1} \leq \frac{|\operatorname{grad} u|^r}{r} + \frac{(r-1) |\operatorname{grad} v|^r}{r}$$

and

$$|\operatorname{grad} v| |\operatorname{grad} u|^{r-1} \leq \frac{|\operatorname{grad} v|^r}{r} + \frac{(r-1)|\operatorname{grad} u|^r}{r}.$$

Therefore f_Z is integrable, by Theorem 1.1 it follows that $\operatorname{div} Z$ is integrable on M and

$$\int_M \operatorname{div} Z \, d\nu_g = 0.$$

On the other hand,

$$\operatorname{div} Z = \alpha' \circ (u - v) h(\operatorname{grad} u, \operatorname{grad} v) + \alpha \circ (u - v) (L_\varphi(u) - L_\varphi(v)).$$

By hypothesis, the second term is ≥ 0 . Thus by Lemma 7.1

$$0 = \int_M \operatorname{div} Z \, d\nu_g \geq \int_M \alpha' \circ (u - v) h(\operatorname{grad} u, \operatorname{grad} v) \, d\nu_g \geq 0.$$

Since $\alpha' \circ (u - v) > 0$ on Ω_A it follows that $h(\operatorname{grad} u, \operatorname{grad} v) = 0$ on Ω_A . Thus $u - v \equiv A$, on Ω_A . Observe that the open set Ω_A is also closed. Since M is connected which implies that $\Omega_A = M$ and then $u - v = A$ on M . \square

An immediate consequence of this theorem is the following corollary.

COROLLARY 7.3. *Assume that (M, g) is a complete Riemannian manifold with infinite volume whose geodesic flow is recurrent with respect to the Liouville measure. If $u \in C^2(M)$ is a subharmonic function on M such that $|\operatorname{grad} u| \in L^2(M)$ and f_X is integrable on SM where $X = \operatorname{grad} u$ then u is constant.*

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